Obtainment of internal labelling operators as broken Casimir operators by means of contractions related to reduction chains in semisimple Lie algebras

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Abstract. We show that the Inönü-Wigner contraction naturally associated to a reduction chain $\mathfrak{s} \supset \mathfrak{s}'$ of semisimple Lie algebras induces a decomposition of the Casimir operators into homogeneous polynomials, the terms of which can be used to obtain additional mutually commuting missing label operators for this reduction. The adjunction of these scalars that are no more invariants of the contraction allow to solve the missing label problem for those reductions where the contraction provides an insufficient number of labelling operators.

1. Introduction

One of the main applications of group theoretical methods to physical problems is related to classification schemes, where irreducible representations of a Lie group have to be decomposed into irreducible representations of a certain subgroup appearing in some relevant reduction chain

$$\left| \begin{array}{ccccc} \mathfrak{s} & \supset & \mathfrak{s}' & \supset & \mathfrak{s}'' \quad ... \\ \downarrow & & \downarrow & & \downarrow \\ [\lambda] & & [\lambda'] & & [\lambda''] \quad ... \end{array} \right\rangle. \tag{1}$$

This is the case for dynamical symmetries used for example in nuclear physics, where one objective of the algebraic model is to describe the Hamiltonian (or mass operator in the relativistic frame) as a function of the Casimir operators of the chain elements. The corresponding energy formulae can the easily deduced from the expectation values in the reduced representations. As example, the Gell-Mann-Okubo mass formula can be derived using this ansatz [1]. In many situations, the labels obtained from the reduction (1) are sufficient to solve the problem, e.g., of we require multiplicity free reductions, as used in $SU(N)$ tumbling gauge models [2] or the interacting boson model [3]. However, often the subgroup does not provide a sufficient number of labels to specify the basis states unambiguously, due to multiplicities greater than one for induced representations. This turns out to be the rule for non-canonical embeddings and generic representations of $\mathfrak{g}$. For some special types, like totally symmetric or

anti-symmetric representations, additional labels are not necessary to solve the problem, and the degeneracies can be solved directly with the available operators.

Many procedures methods have been developed to solve the so-called missing label problem (short MLP), from specific construction of states for the reduction chain to the formal construction of all possible labelling operators using enveloping algebras [4]. The latter procedure allows in theory to find the most general labelling operator, although the effective computation of admissible operators is rather cumbersome. In addition, there is no general criterion to compute the number of operators necessary to generate integrity bases in enveloping algebras.

In the mid seventies, Peccia and Sharp [5] considered an analytic approach to the MLP based on the method used to compute the generalized Casimir invariants of Lie algebras in the commutative frame. This method is very close to the interpretation of Casimir operators as functions that are constant on co-adjoint orbits, and consists essentially on a restriction of this problem. The labelling operators are interpreted as solutions of a certain subsystem of partial differential equations corresponding to the embedded subalgebra. Subgroup scalars are therefore the differential invariants of the subalgebra in a specific realization, and their symmetrization serves to separate multiplicities of reduced representations. Although this approach is, in principle, computationally easier than the pure algebraic approach using operators in the enveloping algebra, the integration of such differential equations is far from being trivial. Moreover, the orthogonality conditions required to the labelling operators, in order to avoid undesirable interactions, must be solved by pure algebraic methods in most cases.

Most of the MLP considered in the literature have been solved or studied from an algebraic or analytic point of view, generally looking for solutions of lowest degrees. In appearance, no attention has been paid to the properties that the embedding imply.[1] The embedding $\mathfrak{s} \supset \mathfrak{s}'$ is conditioned by physical reasons, that is, the choice of the embedding class corresponds to some specific coupling scheme or some relevant internal property that must be emphasized (like angular momentum). Since the embedding determines the branching rules for irreducible representations, it should be expected that labelling operators needed to solve multiplicities are, in some manner, codified by the properties of the symmetry breaking determined by the reduction chain.
When dealing with the MLP algebraically or analytically, it is not immediately clear to which extent we are using the properties of this embedding or the branching rules. Moreover, we can ask whether the obtained labelling operators have some intrinsic or physical meaning at all. Formulated in another way: are the labelling operators of the MLP completely determined by the reduction $\mathfrak{s} \supset \mathfrak{s}'$ (and therefore, by the underlying physics), or are they the result of a formal algebraic/analytic manipulation?

In [7], the missing label problem was approached from a quite general point of view, but having in mind the important fact observed in [8] that symmetry breaking and contractions of Lie algebras have many points in common. This actually is deeply related to the characterization of inhomogeneous algebras obtained from contractions of semisimple algebras [9]. Therefore, any reduction chain $\mathfrak{s} \supset \mathfrak{s}'$ naturally induces a contraction of Lie algebras. It is therefore natural to ask if the operators provided by the contraction $\mathfrak{g}$ (the so-called contracted Casimir operators) can be used to solve the MLP completely. The idea, in a different form, had been used previously for angular momentum subalgebra, and can be seen clearly in the so-called rotor expansion method [4].

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[1] As known, non-equivalent embeddings lead to different branching rules, and therefore to different classification schemes. This is the difference between the Wigner supermultiplet (nuclear LS coupling) and the strange-spin multiplet structure of $\mathfrak{su}(4)$ [6].

The main goal of the general contraction ansatz in labelling problems can be resumed in the following points:

(i) Find a procedure to solve the MLP using explicitly the properties of the embedding $\mathfrak{s} \supset \mathfrak{s}'$ [breaking symmetry $\leftrightarrow$ contractions], that is, without invoking formal operators.

(ii) Justify a natural choice of labelling operators as "broken Casimir operators".

(iii) Find a phenomenological explanation for the non-integer expectation values of labelling operators.

In this first development, only the contracted invariants were used to generate labelling operators. This approach was however sufficient to solve many physically relevant missing label problems, and the result were in perfect harmony with those obtained using other techniques. It was also observed that the method can fail to find a sufficient number of missing operators whenever the identity $\mathcal{N}(\mathfrak{g}) = \mathcal{N}(\mathfrak{s}) = n$ is satisfied. The failure is essentially a consequence of an insufficient number of invariants in the contraction.

The aim of this work is to further develop the contraction procedure, but using not only the contracted invariants, but a certain decomposition induced in the Casimir operators of the contracted algebra, which turn out to be subgroup scalars but no more invariants of the contraction. With this decomposition, we are able to surmount the difficulty for cases with insufficient number of contracted operators. A more interesting consequence of this fact is the possibility of explaining the existence of missing label operators of the same degree, as they have already been constructed in the algebraic frame.

2. Classical Casimir operators

Given a presentation $\mathfrak{s} = \left\{ X_1, .., X_n \mid [X_i, X_j] = C_{ij}^k X_k \right\}$ of a Lie algebra $\mathfrak{s}$ in terms of generators and commutation relations, we are interested in (polynomial) operators $C_p = \alpha^{i_1..i_p} X_{i_1}..X_{i_p}$ in the generators of $\mathfrak{s}$ such that the constraint $[X_i, C_p] = 0$, $(i = 1..n)$ is satisfied. Such an operator necessarily lies in the centre of the enveloping algebra of $\mathfrak{s}$, and is traditionally referred to as Casimir operator. However, in many dynamical problems, the relevant invariant functions are not polynomials, but rational or even trascendental functions (e.g. the inhomogeneous Weyl group). Therefore the approach with the universal enveloping algebra has to be generalized in order to cover arbitrary Lie groups. The most widely used method is the analytical realization. The generators of the Lie algebra $\mathfrak{s}$ are realized by the differential operators $\widehat{X}_i = C_{ij}^k x_k \frac{\partial}{\partial x_j}$, where the x_i are commuting variables associated to each generator X_i. In this approach, a function $F \in C^\infty(\mathfrak{s}')$ is an invariant of $\mathfrak{s}$ if and only if it satisfies the system of PDEs

$$\widehat{X}_i(F) = 0, \quad i = 1..n. \tag{2}$$

Using the symmetrization map $Sym(x_1^{a_1}..x_p^{a_p}) = \frac{1}{k!} \sum_{\sigma \in S_p} x_{\sigma(1)}^{a_1}...x_{\sigma(p)}^{a_p}$, we recover the Casimir operator for polynomial solutions. With this analytical ansatz, it is easily seen that the number of independent solutions is

$$\mathcal{N}(\mathfrak{s}) = \dim \mathfrak{s} - \operatorname{rank} \left[C_{ij}^k x_k \right]. \tag{3}$$

As already commented, one of the main applications of Casimir operators of Lie algebras is the labelling of irreducible representations. In a more general approach, irreducible representations of a Lie algebra $\mathfrak{g}$ can be labelled using the eigenvalues of its generalized Casimir invariants [5]. For each representation, the number of internal labels needed is given by

$$i = \frac{1}{2}(\dim \mathfrak{g} - \mathcal{N}(\mathfrak{g})). \tag{4}$$

For the special case of semisimple Lie algebras, this number is deeply related to the number of positive roots of its complexification. If we use some subalgebra $\mathfrak{g} \supset \mathfrak{h}$ to label the basis states of $\mathfrak{g}$, the embedding provides $\frac{1}{2}(\dim \mathfrak{h} + \mathcal{N}(\mathfrak{h})) + l'$ labels, where l' is the number of invariants of $\mathfrak{g}$ that depend only on variables of the subalgebra $\mathfrak{h}$ [5]. In general, this is still not sufficient to separate multiplicities of induced representations, so that we need to find

$$n = \frac{1}{2}\left(\dim \mathfrak{g} - \mathcal{N}(\mathfrak{g}) - \dim \mathfrak{h} - \mathcal{N}(\mathfrak{h})\right) + l' \tag{5}$$

additional operators, which are commonly called missing label operators. The total number of available operators of this kind is twice the number of needed labels, $m = 2n$. For the case $n > 1$, it remains the problem of determining a set of n mutually commuting operators, as commented before. These operators can be seen to be subgroup scalars, so that the analytical approach is a practical method to find the labelling operators.[2] Considering the realization of $\mathfrak{s} \supset \mathfrak{s}'$ by differential operators indicated in (2), we restrict to the PDEs corresponding to subgroup generators

$$\widehat{X}_i = C_{ij}^k x_k \frac{\partial}{\partial x_j}, \quad 1 \leq i \leq \dim \mathfrak{s}'. \tag{6}$$

Solutions to this system $f(\mathfrak{s}')$ reproduce the differential invariants in $\dim \mathfrak{s}$ dimensions. The total number of solutions of the latter system is given by:

$$\mathcal{N}(f(\mathfrak{s}')) = m + \mathcal{N}(\mathfrak{s}) + \mathcal{N}(\mathfrak{s}') - l'. \tag{7}$$

We observe that (7) refers to the number of functionally independent solutions. Integrity bases, constrained by the coarser algebraic independence, will generally have much more elements than (7), and no general procedure is known to compute its dimension. In addition, to be useful as labelling operators, two (symmetrized) solutions F_1, F_2 of system (6) must satisfy the orthogonality condition $[F_1, F_2] = 0$ and commute with the Casimir operators C_i and D_j of $\mathfrak{s}$ and the subalgebra $\mathfrak{s}'$. Therefore the set of commuting operators $\{C_i, D_j, F_k\}$ serves to label the states unambiguously.

It was proved in [7] that any reduction chain $\mathfrak{s} \supset \mathfrak{s}'$ is naturally associated to an Inönü-Wigner contraction $\mathfrak{s} \rightsquigarrow \mathfrak{g} = \mathfrak{s}' \overrightarrow{\oplus}_R (\dim \mathfrak{s} - \dim \mathfrak{s}') L_1$, where the representation R of $\mathfrak{s}'$ satisfies the constraint[3]

$$\mathrm{ad}\mathfrak{s} = \mathrm{ad}\mathfrak{s}' \oplus R. \tag{8}$$

The contraction is easily seen to be defined by the non-singular transformations

$$\Phi_t(X_i) = \begin{cases} X_i, & 1 \leq i \leq s \\ \frac{1}{t} X_i, & s+1 \leq i \leq n \end{cases}, \tag{9}$$

where $\{X_1, .., X_s, X_{s+1}, .., X_n\}$ is a basis of $\mathfrak{s}$ such that $\{X_1, .., X_s\}$ generates the subalgebra $\mathfrak{s}'$ and $\{X_{s+1}, .., X_n\}$ spans the representation space R. If both $\mathfrak{s}$ and $\mathfrak{s}'$ are semisimple, the contraction is isomorphic to an inhomogeneous Lie algebra with Levi decomposition $\mathfrak{g} = \mathfrak{s}' \overrightarrow{\oplus}_R (\dim \mathfrak{s} - \dim \mathfrak{s}') L_1$. Since it satisfies the condition $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, this algebra admits a fundamental basis of invariants consisting of Casimir operators. Moreover, by the contraction we have the inequality $\mathcal{N}(\mathfrak{s}) \leq \mathcal{N}(\mathfrak{g})$. The system of PDEs corresponding to this contraction can be divided into two parts:

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[2] In many cases, the degeneracies of the reduction chain can be solved without using the subgroup scalars, or are determined by a specifical ansatz adapted to the involved groups.

[3] Actually, the branching rule depends on a numerical index j_f that characterizes the embedding class.

$$\widehat{X}_i F = C_{ij}^k x_k \frac{\partial F}{\partial x_j} = 0, \qquad\qquad 1 \le i \le s, \tag{10}$$

$$\widehat{X}_{s+i} F = C_{s+i,j}^{s+k} x_{s+k} \frac{\partial F}{\partial x_j} = 0, \quad 1 \le i, k \le n-s, 1 \le j \le s. \tag{11}$$

The subsystem (10) corresponds to the generators of $\mathfrak{s}'$ realized as subalgebra of $\mathfrak{s}$, while the remaining equations (11) describe the representation. Written in matrix form, the system is given by

$$
\begin{pmatrix}
0 & \cdots & C_{1s}^k x_k & C_{1,s+1}^k x_k & \cdots & C_{1,n}^k x_k \\
\vdots & & \vdots & \vdots & & \vdots \\
-C_{1s}^k x_k & \cdots & 0 & C_{s,s+1}^k x_k & \cdots & C_{s,n}^k x_k \\
-C_{s,s+1}^k x_k & \cdots & -C_{s,s+1}^k x_k & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots \\
-C_{1n}^k x_k & \cdots & -C_{s,n}^k x_k & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
\partial_{x_1} F \\
\vdots \\
\partial_{x_s} F \\
\partial_{x_{s+1}} F \\
\vdots \\
\partial_{x_n} F
\end{pmatrix} = 0.
$$

Since the first s first rows reproduce exactly the system of PDEs needed to compute the missing label operators, we conclude that any invariant of $\mathfrak{g}$ is a candidate for missing label operator. In view of this situation, the following questions arise naturally:

(i) Do polynomial functions of the invariants of these algebras suffice to determine n mutually orthogonal missing label operators?

(ii) Can all available operators found by this procedure?

The answer to the first equation is in the affirmative for those cases where the contraction provides a number of independent invariants exceeding the number of needed labelling operators. It fails when these two quantities coincide, which suggests that the procedure has to be refined. The answer to the second question is generally in the negative (see e.g. the algebras with one missing label), although it cannot be excluded that with the refinement proposed in this work we are able to recover a complete set of independent labelling operators for some special reductions. In most cases, only half of the available operators should be expected, since all operators obtained are the result, in some sense, of "breaking" the original Casimir operators. Whether a further refinement allows to obtain additional labelling operators that are independent remains for the moment an unanswered question.

3. Decomposition of Casimir operators

In this section we prove that the contraction induced by the reduction chain induces a decomposition of the corresponding Casimir operators of $\mathfrak{s}$, which allow, among other properties, to determine the invariants of the contraction $\mathfrak{g}$. However, other terms are also relevant for the missing label problem, and constitute the solution to the problem pointed out in [7] when the number of invariants of the contraction is not sufficient. These additional terms do not constitute invariants of the contraction, and where therefore not considered in [7].

We briefly recall the definition of contracted invariants. Since classical Inönü-Wigner contractions are the only type of contractions needed for the labelling problem, we can restrict ourselves to this case [10]. Let $C_p(X_1, ..., X_n) = \alpha^{i_1 \dots i_p} X_{i_1} ... X_{i_p}$ be a pth-order Casimir operator of $\mathfrak{s}$. Then the transformed invariant takes the form

$$F(\Phi_t(X_1), .., \Phi_t(X_n)) = t^{n_{i_1} + \dots + n_{i_p}} \alpha^{i_1 \dots i_p} X_{i_1} ... X_{i_p}, \tag{12}$$

where $n_{i_j} = 0, 1$. Taking the maximal power in t,

$$M = \max\left\{n_{i_1} + ... + n_{i_p} \quad | \quad \alpha^{i_1..i_p} \neq 0\right\}, \tag{13}$$

the limit

$$F'(X_1, .., X_n) = \lim_{t \to \infty} t^{-M} F(\Phi_t(X_1), ..., \Phi_t(X_n))$$

$$= \sum_{n_{i_1} + ... + n_{i_p} = M} \alpha^{i_1...i_p} X_{i_1}...X_{i_p}$$

provides a Casimir operator of degree p of the contraction $\mathfrak{g}'$. Now, instead of extracting only the term with the highest power of t, we consider the whole decomposition

$$C_p = t^M C'_p + \sum_\alpha t^\alpha \Phi_\alpha + \Phi_0, \tag{14}$$

where $\alpha < M \leq p$ and Φ_0 is a function of the Casimir operators of the subalgebra $\mathfrak{s}'$ (these generators have not been re-scaled). It is straightforward to verify that C'_p is not only an invariant of the contraction $\mathfrak{g}$, but also a solution to the MLP.[4] This first term was central to the argumentation in [7], and allowed to obtain commuting sets of labelling operators. However, the remaining terms can also be individually considered as candidates for labelling operators, as states the following

Proposition 1 *The functions Φ_α are solutions of the missing label problem, that is, they satisfy the system*

$$\widehat{X}_i \Phi_\alpha = C_{ij}^k x_k \frac{\partial \Phi_\alpha}{\partial x_j} = 0, \quad 1 \leq i \leq s. \tag{15}$$

Proof. First of all, the decomposition (14) tells that the Casimir operator C_p can be rewritten as a sum of homogeneous polynomials C'_p, Φ_α with the property that C'_p is of homogeneity degree $p - M$ in the variables $\{x_1, .., x_s\}$ associated to subalgebra generators and degree M in the remaining variables $\{x_{s+1}, .., x_n\}$. Accordingly, any Φ_α is of degree $p - \alpha$ in the variables $\{x_1, .., x_s\}$ and α in the $\{x_{s+1}, .., x_n\}$. We denote this by saying that these functions are of bi-degree $(p - \alpha, \alpha)$.

Now the equations (10) corresponding to subalgebra generators remain unaltered by the contraction procedure, since the re-scaling of generators does not affect them.[5] Thus for any $1 \leq i \leq s$ and any homogeneous polynomial Ψ of bi-degree $(p - q, q)$ we obtain

$$\widehat{X}_i \Psi = C_{ij}^k x_k \frac{\partial \Psi}{\partial x_j} + C_{ij+s}^{k+s} x_{k+s} \frac{\partial \Psi}{\partial x_j}, \tag{16}$$

and the result is easily seen to be again a polynomial with the same bi-degree. This means that evaluating $C_p = t^M C'_p + \sum_\alpha t^\alpha \Phi_\alpha + \Phi_0$ is a sum of polynomials of different bi-degree, and since C_p is a Casimir operators, the only possibility is that each term is a solution of the system. We thus conclude that the Φ_α are solutions of (10). ∎

The first question that arises from decomposition (14) is how many independent additional solutions we obtain. Since all Φ_α together sum the Casimir operator, some dependence relations must exist.

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[4] For $t = 1$, equation (14) shows how the Casimir operator decomposes into homogeneous polynomials in the variables of the subalgebra and the complementary space over the original basis.

[5] On the contrary, for the remaining equations the differential operators of the generators corresponding to the representation have been modified, and the equations are dependent on t.

Lemma 1 *Let C_p be a Casimir operator of $\mathfrak{s}$ of order p. Suppose that*

$$C_p = \Phi_{(p-\alpha_1,\alpha_1)} + ... + \Phi_{(p-\alpha_q,\alpha_q)}, \quad 0 \leq \alpha_i < \alpha_{i+1} \leq p \tag{17}$$

is the decomposition of C_p into homogeneous polynomials of bi-degree (p,q).

(i) If $\Phi_{(0,p)} \neq 0$, then at most $q-2$ polynomials $\Phi_{(p-\alpha_j,\alpha_j)}$ are functionally independent on the Casimir operators of $\mathfrak{s}$ and $\mathfrak{s}'$.

(ii) If $\Phi_{(0,p)} = 0$, then at most $q-1$ polynomials $\Phi_{(p-\alpha_j,\alpha_j)}$ are functionally independent on the Casimir operators of $\mathfrak{s}$ and $\mathfrak{s}'$.

The proof follows at once observing that $\Phi_{(0,p)}$ is a function of the Casimir operators of the subalgebra $\mathfrak{s}'$. The independence on the Casimir operators of $\mathfrak{s}'$ does not imply in general that the $\Phi_{(p-\alpha,\alpha)}$ obtained are all functionally independent between themselves. The number of independent terms depends also on the representation R induced by the reduction [11]. In any case, however, at least one independent term is obtained for any Casimir operator of degree at least three. For the special case of $n = 1$ labelling operator, two terms independent on the Casimir operators were found, which allowed to select one as the labelling operator [7]. Once a set of functionally independent solutions to system (10) has been chosen (including the Casimir operators), the first part of the labelling problem is solved. Now, if we want to obtain a set of commuting operators, we have to look for all commutators among the symmetrized operators $\Phi_{(p-\alpha_j,\alpha_j)}$. We denote by $\Phi_{(p-\alpha_j,\alpha_j)}^{symm}$ the symmetrized polynomial. Then $\left[\Phi_{(p-\alpha_j,\alpha_j)}^{symm}, \Phi_{(q-\alpha_k,\alpha_k)}^{symm} \right]$ is a homogeneous polynomial of degree $p + q - 1$, and also constitutes a missing label operator. Actually this brackets is expressible as sum of polynomials of different bi-degree, and these terms constitute themselves labelling operators [12]. A procedure to solve the missing label problem can thus be resumed in the following steps:

- Decompose the Casimir operators of $\mathfrak{s}$ of degree $p \geq 3$ with respect to the associated contraction.
- Determine the commutator of all symmetrized polynomials $\Phi_{(p-\alpha_j,\alpha_j)}^{symm}$ with $\alpha_j \neq 0$.
- From those commuting operators, extract n operators that are functionally independent from the Casimir operators of $\mathfrak{s}$ and the subalgebra $\mathfrak{s}'$.

In general, the second step is reduced to pure computation. There is no simple procedure to determine whether two missing label operators are mutually orthogonal, although various symbolic routines have been developed to compute these brackets (see e.g. [13]). In some special circumstances, however, the decomposition (14) can provide orthogonality without being forced to compute the brackets. If for a specific MLP it is known that no solutions of bi-degree (r, s) exists for some fixed $r + s = p + q$, and if we have two labelling operators such that $\left[\Phi_{(p-\alpha_j,\alpha_j)}^{symm}, \Phi_{(q-\alpha_k,\alpha_k)}^{symm} \right]$ is a sum of polynomials of bi-degree (s, r), then the commutation follows at once. This idea was first explored systematically in [12]. We remark that in the commutative frame, it would suffice to show that no polynomial function of bi-degree (r, s) is a solution to subsystem (10).[6]

4. Examples
In this section we show how the decomposition of Casimir operators of higher order provide solutions to missing label problem that could not be solved completely by only using the contraction, or for which no proposed set of labelling operators has been computed yet. We insist on the fact that the main difficulty in the formal approach to the MLP resides in obtaining a sufficient number of (functionally) independent labelling operators, from which a commuting set can be extracted.

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[6] The labelling operator in the enveloping algebra of $\mathfrak{s}$ follows as the symmetrized form of such functions.

4.1. $G_2 \supset \mathfrak{su}(2) \times \mathfrak{su}(2)$

This chain was indicated in [7] to give an insufficient number of labelling operators when only the contraction invariants are considered. Actually, in this case we have $n = \frac{1}{2}(14 - 2 - 6 - 2) = 2$ labelling operators, and the inhomogeneous contraction $G_2 \rightsquigarrow (\mathfrak{su}(2) \times \mathfrak{su}(2)) \overrightarrow{\oplus}_R 8L_1$ preserves the number of invariants. This means that we would only obtain one additional operator, since the (contracted) operator of order two is of no use. Now, the method failed because it did not take into account the decomposition of the sixth order operator into homogeneous polynomials of bi-degree (p, q) in the variables. We show that, with this decomposition, we obtain a complete solution to the MLP related to the chain $G_2 \supset \mathfrak{su}(2) \times \mathfrak{su}(2)$. To this extent, we choose the same tensor basis used in [14] consisting of the generators $\{j_0, j_\pm, k_0, k_\pm, R_{\mu,\nu}\}$ with $\mu = \pm\frac{3}{2}, \pm\frac{1}{2}$, $\nu = \pm\frac{1}{2}$. The generators $R_{\mu,\nu}$ are related to an irreducible tensor representation R of $\mathfrak{su}(2) \times \mathfrak{su}(2)$ of order eight. In this case, the contraction $G_2 \rightsquigarrow (\mathfrak{su}(2) \times \mathfrak{su}(2)) \overrightarrow{\oplus}_R 8L_1$ is obtained considering the transformations:

$$j'_0 = j_0, \ j'_\pm = j_\pm, \ k'_0 = k_0, k'_\pm = k_\pm, \ R'_{\mu,\nu} = \frac{1}{t} R_{\mu,\nu}.$$

If we decompose now the Casimir operators C_2 and C_6 over the transformed basis, we obtain the following decomposition

$$\begin{aligned}
C_2 &= t^2 C_{(2,0)} + C_{(0,2)}, \\
C_6 &= t^6 C_{(6,0)} + t^4 C_{(4,2)} + t^2 C_{(2,4)} + C_{(0,6)},
\end{aligned} \tag{18}$$

where $C_{(0,2)}, C_{(0,6)}$ are functions of the Casimir operators of $\mathfrak{su}(2) \times \mathfrak{su}(2)$. Since $C_{(2,0)}$ is functionally dependent on the invariants of G_2 and $\mathfrak{su}(2) \times \mathfrak{su}(2)$, it is not further useful. Now it can be verified that

$$\frac{\partial\left(C_2, C_6, C_{21}, C_{22}, C_{(2,4)}, C_{(4,2)}\right)}{\partial\left(k_0, k_-, j_0, j_+, R_{\frac{3}{2}, \frac{1}{2}}, R_{-\frac{3}{2}, \frac{1}{2}}\right)} \neq 0, \tag{19}$$

where C_{21} and C_{22} are the quadratic Casimir operators of $\mathfrak{su}(2) \times \mathfrak{su}(2)$. This provides us with six independent operators. A long and tedious computation, due to the quite high number of terms before and after symmetrization, shows moreover that the chosen operators commute:

$$\begin{aligned}
\left[C_i, C_{(2,4)}\right] &= \left[C_i, C_{(4,2)}\right] = 0, \quad i = 2, 6 \\
\left[C_{(4,2)}, C_{(2,4)}\right] &= 0.
\end{aligned} \tag{20}$$

Therefore the set $\left\{C_2, C_6, C_{21}, C_{22}, C_{(2,4)}, C_{(4,2)}\right\}$ can be taken to solve the labelling problem.

It should be remarked that a direct comparison with the operators obtained in [14] is quite difficult, for various reasons. At first, there the scalars in the enveloping algebra were considered, not symmetrizations of functions, which implies that lower order terms where considered when explicitly indicating the labelling operators. On the other hand, we have only distinguished the bi-degree, that is, the degree of the polynomials in the variables of the $\mathfrak{su}(2) \times \mathfrak{su}(2)$ subalgebra and the tensor representation R, while in [14] the order with respect to any of the copies of $\mathfrak{su}(2)$ was considered, resulting in operators labelled with three indices. Therefore the operators $C_{(p,q)}$ considered here correspond to the sum of several scalars there. In addition, our solution contains the term $C^{(114)}$ excluded in [14],[7] confirming that the pair of commuting operators obtained above is different from that found previously. We also remark that a further distinction of the degrees of the polynomials $\Phi^{symm}_{(a,b)}$ in the variables of the $\mathfrak{su}(2)$ copies is not possible due to the contraction.

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[7] This is a scalar having degree one in each of the copies of $\mathfrak{su}(2)$ and four in the $R_{\mu,\nu}$ generators.

4.2. The chain $\mathfrak{sp}(6) > \mathfrak{su}(3) \times \mathfrak{u}(1)$

The unitary reduction of the symplectic Lie algebra of rank three has found ample applications in the nuclear collective model [15]. In this case, nuclear states are classified by means of irreducible representations of $\mathfrak{sp}(6)$ reduced with respect to the unitary subalgebra $\mathfrak{su}(3) \times \mathfrak{u}(1)$. Since the induced representations are not multiplicity free, we have to add $n = 3$ labelling operators to distinguish the states. Generating functions for this chain were studied in [16], but without obtaining explicitly the three required operators. In this section, we will determine a commuting set of labelling operators that solves the MLP for this reduction. As we shall see, this case cannot be solved using only the invariants of the associated contraction.

We will use the Racah realization for the symplectic Lie algebra $\mathfrak{sp}(6, \mathbb{R})$. We consider the generators $X_{i,j}$ with $-3 \leq i, j \leq 3$ satisfying the condition

$$X_{i,j} + \varepsilon_i \varepsilon_j X_{-j,-i} = 0, \tag{21}$$

where $\varepsilon_i = \operatorname{sgn}(i)$. Over this basis, the brackets are given by

$$[X_{i,j}, X_{k,l}] = \delta_{jk} X_{il} - \delta_{il} X_{kj} + \varepsilon_i \varepsilon_j \delta_{j,-l} X_{k,-i} - \varepsilon_i \varepsilon_j \delta_{i,-k} X_{-j,l}, \tag{22}$$

where $-3 \leq i, j, k, l \leq 3$. The three Casimir operators C_2, C_4, C_6 of $\mathfrak{sp}(6, \mathbb{R})$ are easily obtained as the coefficients of the characteristic polynomial

$$|A - T \operatorname{Id}_6| = T^6 + C_2 T^4 + C_4 T^2 + C_6, \tag{23}$$

where

$$A = \begin{pmatrix} x_{1,1} & x_{2,1} & x_{3,1} & -Ix_{-1,1} & -Ix_{-1,2} & -Ix_{-1,3} \\ x_{1,2} & x_{2,2} & x_{3,2} & -Ix_{-1,2} & -Ix_{-2,2} & -Ix_{-2,3} \\ x_{1,3} & x_{2,3} & x_{3,3} & -Ix_{-1,3} & -Ix_{-2,3} & -Ix_{-3,3} \\ Ix_{1,-1} & Ix_{1,-2} & Ix_{1,-3} & -x_{1,1} & -x_{1,2} & -x_{1,3} \\ Ix_{1,-2} & Ix_{2,-2} & Ix_{2,-3} & -x_{2,1} & -x_{2,2} & -x_{2,3} \\ Ix_{1,-3} & Ix_{2,-3} & Ix_{3,-3} & -x_{3,1} & -x_{2,3} & -x_{3,3} \end{pmatrix}. \tag{24}$$

The symmetrized operators give the usual polynomials in the enveloping algebra. Since the unitary algebra $\mathfrak{u}(3)$ is generated by $\{X_{i,j} | 1 \leq i, j \leq 3\}$, in order to write $\mathfrak{sp}(6, \mathbb{R})$ in a $\mathfrak{su}(3) \times \mathfrak{u}(1)$ basis, it suffices to replace the diagonal operators $X_{i,i}$ by suitable linear combinations. Taking $H_1 = X_{1,1} - X_{2,2}$, $H_2 = X_{2,2} - X_{3,3}$ and $H_3 = X_{1,1} + X_{2,2} + X_{3,3}$ we obtain the Cartan subalgebra of $\mathfrak{su}(3)$, while H_3 commutes with all $X_{i,j}$ with positive indices i, j. The invariants over this new basis are simply obtained replacing the $x_{i,i}$ by the corresponding linear combinations of h_i. The contraction $\mathfrak{sp}(6) \rightsquigarrow (\mathfrak{su}(3) \times \mathfrak{u}(1)) \overrightarrow{\oplus}_R 12 L_1$, where R is the complementary to $(\operatorname{ad}(\mathfrak{su}(3) \otimes (1))$ in the adjoint representation of $\mathfrak{sp}(6)$:[8]

$$\operatorname{ad}\mathfrak{sp}(6) = (\operatorname{ad}\mathfrak{su}(3) \otimes (1)) \oplus R.$$

The contraction is determined by the transformations

$$H_i' = H_i, \ X_{i,j}' = X_{i,j}, \ X_{-i,j}' = \frac{1}{t} X_{-i,j}, \ X_{i,-j}' = \frac{1}{t} X_{i,-j}, \quad 1 \leq i, j \leq 3. \tag{25}$$

The contraction $(\mathfrak{su}(3) \times \mathfrak{u}(1)) \overrightarrow{\oplus}_R 12 L_1$ satisfies $\mathcal{N} = 3$, thus has 3 Casimir operators that can be obtained as contraction of C_2, C_4, C_6. Note however that $n = 3$, thus the invariants of the contraction will provide at most two independent missing label operators. This means that using

---

[8] More precisely, R decomposes into a sextet and antisextet with $\mathfrak{u}(1)$ weight ± 1 and a singlet with $\mathfrak{u}(1)$ weight 1.

only the contraction, we cannot solve the MLP for this chain. In order to find a third labelling operator, we have to consider the decomposition of the fourth and sixth order Casimir operators of $\mathfrak{sp}(6)$. Over the preceding transformed basis we obtain:

$$
\begin{aligned}
C_4 &= t^4 C_{(4,0)} + t^2 C_{(2,2)} + C_{(0,4)}, \\
C_6 &= t^6 C_{(6,0)} + t^4 C_{(4,2)} + t^2 C_{(2,4)} + C_{(0,6)},
\end{aligned}
\tag{26}
$$

where $C_{(k,l)}$ denotes a homogeneous polynomial of k in the variables of R and degree l in the variables of the unitary subalgebra. The $C_{(0,k)}$ are functions of the Casimir operators of $\mathfrak{su}(3) \times \mathfrak{u}(1)$, and therefore provide no labelling operators. We remark that, before symmetrization, $C_{(2,2)}$ has 126 terms, $C_{(2,4)}$ 686 terms, and $C_{(4,2)}$ 444 terms. The symmetrized operators $C_{(2,2)}, C_{(4,2)}$ and $C_{(2,4)}$ can be added to the Casimir operators of $\mathfrak{sp}(6)$ and the subalgebra $\mathfrak{su}(3) \times \mathfrak{u}(1)$, and the 9 operators can be seen to be (functionally) independent.

$$
\begin{aligned}
\left[C_i, C_{(2,2)}\right] &= \left[C_i, C_{(4,2)}\right] = \left[C_i, C_{(2,4)}\right] = 0, \qquad i = 2,4,6. \\
\left[C_{(2,2)}, C_{(4,2)}\right] &= \left[C_{(2,2)}, C_{(2,4)}\right] = \left[C_{(2,4)}, C_{(4,2)}\right] = 0.
\end{aligned}
\tag{27}
$$

4.3. Applications to the conformal algebra

Among the many important problems in Physics where the conformal group $SO(2,4)$ plays an impotant role, like the dynamical non-invariance group of hydrogen-like atoms, the application to the periodic charts of neutral atoms in ions was first considered in [17]. This direction was followed to classify chemical elements by various authors [18]. More recently, the conformal group and its invariants are in the centre of the more ambitious program KGR, in order to obtain quantitative predictions of the periodic table of elements [19, 20]. To this extent, the set formed by the three Cartan generators and the Casimir operators (of degrees $2, 3$ and 4), which commute between themselves, can be used to label certain physical properties. However, as noted by Racah [21], this set is still not sufficient for classification purposes. We have to add three additional operators[9] to obtain a complete set of commuting operators that solve labelling problems. This follows at once if we consider the missing label problem for the Cartan subalgebra. In this case

$$
n = \frac{1}{2} \left(15 - 3 - 3 - 3 \right) = 3.
$$

Therefore the Racah operators can be identified with labelling operators for the reduction chain determined by the Cartan subalgebra. To exemplify the procedure, we compute the Racah operators for the conformal algebra. We use the fact that it is isomorphic to the Lie algebra $\mathfrak{su}(2,2)$. We start from the the $\mathfrak{u}(2,2)$-basis formed by the operators $\{E_{\mu\nu}, F_{\mu\nu}\}_{1 \leq \mu,\nu \leq p+q=n}$ with the constraints

$$
\begin{aligned}
E_{\mu\nu} + E_{\nu\mu} &= 0, \; F_{\mu\nu} - F_{\nu\mu} = 0, \\
g_{\mu\mu} &= \left((1,1,-1,-1\right).
\end{aligned}
$$

The brackets are then given by

$$
[E_{\mu\nu}, E_{\lambda\sigma}] = g_{\mu\lambda} E_{\nu\sigma} + g_{\mu\sigma} E_{\lambda\nu} - g_{\nu\lambda} E_{\mu\sigma} - g_{\nu\sigma} E_{\lambda\mu} \tag{28}
$$

$$
[E_{\mu\nu}, F_{\lambda\sigma}] = g_{\mu\lambda} F_{\nu\sigma} + g_{\mu\sigma} F_{\lambda\nu} - g_{\nu\lambda} F_{\mu\sigma} - g_{\nu\sigma} F_{\lambda\mu} \tag{29}
$$

$$
[F_{\mu\nu}, F_{\lambda\sigma}] = g_{\mu\lambda} E_{\nu\sigma} + g_{\nu\lambda} E_{\mu\sigma} - g_{\nu\sigma} E_{\lambda\mu} - g_{\mu\sigma} E_{\lambda\nu} \tag{30}
$$

To recover the conformal algebra, we take the Cartan subalgebra spanned by the vectors $H_\mu = g_{\mu+1,\mu+1} F_{\mu\mu} - g_{\mu\mu} F_{\mu+1,\mu+1}$ for $\mu = 1..3$. The centre of $\mathfrak{u}(p,q)$ is obviously generated by $g^{\mu\mu} F_{\mu\mu}$.

---

[9] If r denotes the dimension of a semisimple Lie algebra $\mathfrak{s}$ and l its rank, the number $f = \frac{1}{2}(r - 3l) = 3$ is usually referred to as the Racah number.

Proposition 2 *A maximal set of independent Casimir invariants of* $\mathfrak{su}(2,2)$ *is given by the coefficients* C_k *of the characteristic polynomial* $|IA - \lambda\mathrm{Id}_N| = \lambda^4 + \sum_{k=2}^{4} D_k\lambda^{4-k}$, *where*

$$
A = \begin{pmatrix}
-I(\frac{3}{4}h_1 - \frac{1}{2}h_2 + \frac{1}{4}h_3) & -e_{12} - If_{12} & e_{13} + If_{13} & e_{14} + If_{14} \\
e_{12} - If_{12} & I(\frac{1}{4}h_1 + \frac{1}{2}h_2 - \frac{1}{4}h_3) & e_{23} + If_{23} & e_{24} + If_{24} \\
e_{13} - If_{13} & e_{23} - If_{23} & I(\frac{1}{4}h_1 - \frac{1}{2}h_2 - \frac{1}{4}h_3) & e_{34} + If_{34} \\
e_{14} - If_{14} & e_{24} - If_{24} & -e_{34} + If_{34} & I(\frac{1}{4}h_1 - \frac{1}{2}h_2 + \frac{3}{4}h_3)
\end{pmatrix}.
\tag{31}
$$

The classical Casimir operators are obtained symmetrizing the functions C_k. In order to compute the Racah operators, we consider the MLP for the chain $\mathfrak{h} \subset \mathfrak{su}(2,2)$, where $\mathfrak{h}$ denotes the Cartan subalgebra. The corresponding contraction[10] is defined by the non-singular transformations

$$
H_i' = \frac{1}{t}H_i, \quad i = 1, 2, 3.
$$

According to this contraction, the Casimir operators decompose as follows:

$$
\begin{aligned}
C_2 &= t^2 C_{(2,0)} + C_{(0,2)}, \\
C_3 &= t^3 C_{(3,0)} + t^2 C_{(2,1)} + C_{(0,3)}, \\
C_4 &= t^4 C_{(4,0)} + t^3 C_{(3,1)} + t^2 C_{(2,2)} + C_{(0,4)},
\end{aligned}
\tag{32}
$$

where the $C_{(0,i)}$ are functions of h_1, h_2, h_3. The functions I_{ij} are all solutions to the MLP. In order to complete the set of orthogonal operators $\{H_1, H_2, H_3, C_2, C_3, C_4\}$ with three mutually commuting labelling operators, we first extract those triples that are functionally independent from the Casimir operators of $\mathfrak{su}(2,2)$ and the h_i. We can take for example $C_{(3,0)}, C_{(4,0)}, C_{(3,1)}$. Since

$$
\frac{\partial\big(H_1, H_2, H_3, C_2, C_3, C_4, C_{(3,0)}, C_{(4,0)}, C_{(3,1)}\big)}{\partial(h_1, h_2, h_3, e_{12}, e_{13}, e_{14}, f_{23}, f_{24}, f_{34})} \neq 0,
\tag{33}
$$

these operators are independent. A somewhat more laborious computation shows that the symmetrized forms of $C_{(2,1)}, C_{(4,0)}, C_{(3,1)}$ satisfy the commutators

$$
\begin{aligned}
\big[C_i, C_{(3,0)}\big] &= \big[C_i, C_{(4,0)}\big] = \big[C_i, C_{(3,1)}\big] = 0, \qquad i = 1, 2, 3. \\
\big[C_{(3,0)}, C_{(4,0)}\big] &= \big[C_{(3,0)}, C_{(3,1)}\big] = \big[C_{(4,0)}, C_{(3,1)}\big] = 0.
\end{aligned}
\tag{34}
$$

Since the $C_{(i,j)}$ are solutions to the MLP, they commute with any function of the generators H_i. In contrast to the previous cases, the length of the labelling operators never exceeds 70 terms before symmetrization. In conclusion, the set $\big\{H_1, H_2, H_3, C_2, C_3, C_4, C_{(3,0)}, C_{(4,0)}, C_{(3,1)}\big\}$ is complete formed by commuting operators. We observe that linear combinations of the three Racah operators are of potential use to describe chemical and physical properties like ionization energy, atomic volume or magnetic properties. Since this identification relies heavily on experimental data [18], it remains to compute the corresponding eigenvalues for irreducible representations (IRREPs) of $\mathfrak{su}(2,2)$, which constitutes a quite hard numerical problem. This task is in progress.

5. Conclusions

The method of contraction is useful to solve the MLP when the number of invariants of the contraction associated to the reduction chain $\mathfrak{s} \supset \mathfrak{s}'$ exceeds the number of needed labelling operators. In the case where the invariants of the inhomogeneous contraction do not suffice to

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[10] In this case, the contraction is no more an inhomogeneous Lie algebra. The procedure remains however valid, which suggests that it could also be valid for non-semisimple algebras.

find a complete solution of the missing label problem, it is expectable that labelling operators of the same degree appear. This suggests that further terms of the Casimir operators of $\mathfrak{s}$ that disappear during the contraction can be useful to complete the set of missing label operators. We have shown that the contraction induces a decomposition of the Casimir operators, the terms of which are all solutions to the MLP. From these terms a set of n independent labelling operators can be extracted, reducing the problem to determine which combinations are mutually orthogonal. In this sense, the method proposed in [7] is a first approximation to solve the MLP using the properties of reduction chains, which however turns out to be useful in most practical cases. The bi-degree of the Casimir operators of a Lie algebra with respect to the variables associated to the generators of a subalgebra are therefore a relevant tool to obtain and classify these labelling operators, although further distinction of terms, for example when the subalgebra consists of various copies, is also convenient to deduce additional operators.[11] This subdivision cannot however be deduced from the contraction, since all generators of the subalgebra play the same role.

Some important aspects of the decomposition method of Casimir operators based on the contractions and its use in labelling problems are specially emphasized:

- The solutions provide a "natural" choice for the labelling operators. Their interpretation as "broken" Casimir operators confers them a certain physical meaning, in contrast to operators obtained by pure algebraic means, where the physical interpretation of the operator is often not entirely clear.

- The decomposition provides also a consistent explanation to the question why a number of reduction chains give labelling operators of the same degree. This fact is directly related to an insufficient number of invariants in the contraction associated to the chain.

- This could probably explain why the eigenvalues of such labelling operators are not integers, as already indicated by Racah [21]. It follows from the decomposition that the eigenvalues of the labelling operators contribute to the eigenvalues of the Casimir operators. In this context, the interpretation of a labelling operator as "broken" Casimir operator leads to the idea of "broken" integer eigenvalues.

Some questions still remain open, namely, whether there exist reductions $\mathfrak{s} \supset \mathfrak{s}'$ for which the method followed here provides all available labelling operators. An answer in this direction implies to find the general solution to the MLP for each considered chain. Nowadays, only for a few number of algebras these computations have been carried out completely [4, 22]. A complete study of all physically relevant reduction chains involving simple Lie algebras up to some fixed rank would certainly provide new insights to this problem. On the other hand, in can also not be excluded that for reduction chains with a great number of labelling operators, the terms of the decomposition are not sufficient to construct a set of independent labelling operators. To which extent the invariants of the contraction not appearing as contracted operators play a role must still be analyzed.[12]

Another problem, still in progress, is to obtain complete sets of commuting operators for all simple Lie algebras, using the MLP determined by the Cartan subalgebra. The commented application to the periodic charts of atoms in only one of the problems where this special type of reductions have been shown to be of interest in developing algebraic models in molecular physics or nuclear spectroscopy [23].

---

[11] This turns out to be the case for the chain $\mathfrak{su}(4) \supset \mathfrak{su}(2) \times \mathfrak{su}(2)$ [4].

[12] Such situations appear, e.g., considering a simple algebra of high rank and regular subalgebras of low rank. If the induced representation (8) contains copies of the trivial representation, then the generators associated to these will play the role of labelling operators. It happens moreover that these generators cannot be obtained contracting the Casimir operators of $\mathfrak{s}$. This situation is however unlike to appear in some physically interesting case.

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